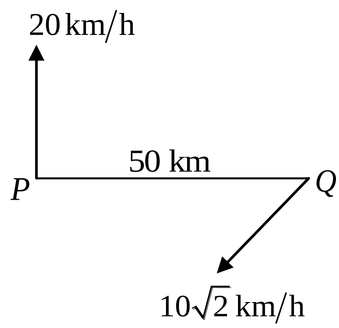
Applied Maths Induction Workshop 2 – Relative Motion – Solutions

2009 – Ordinary Level – Question 2

A ship P is moving north at a constant speed of $20\,\text{km/h}$.

Another ship Q is moving south-west with a constant speed of $10\sqrt{2}\,\text{km/h}$.

At a certain instant, P is positioned 50 km due west of Q.



Find (i) the velocity of P in terms of $\vec{i}$ and $\vec{j}$.

(ii) the velocity of Q in terms of $\vec{i}$ and $\vec{j}$.

(iii) the velocity of P relative to Q in terms of $\vec{i}$ and $\vec{j}$.

(iv) the shortest distance between P and Q in the subsequent motion.

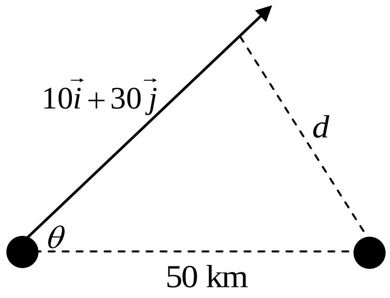
Solution

(i) $\overrightarrow{v_P} = 20\vec{j}$

(ii) $\overrightarrow{v_Q} = -10\sqrt{2}\cos 45^\circ \vec{i} - 10\sqrt{2}\sin 45^\circ \vec{j} = -10\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\vec{i} - 10\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)\vec{j} = -10\vec{i} - 10\vec{j}$

(iii) $\overrightarrow{v_{PQ}} = \overrightarrow{v_P} - \overrightarrow{v_Q} = 20\vec{j} - \left(-10\vec{i} - 10\vec{j}\right) = 10\vec{i} + 30\vec{j}$

(iv)



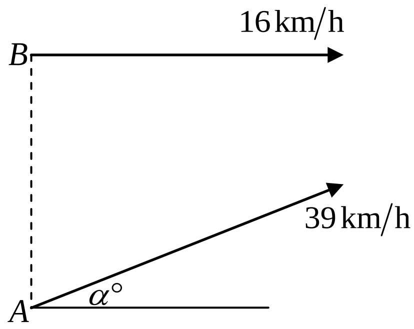
$\tan\theta = \frac{30}{10} = 3 \Rightarrow \sin\theta = \frac{3}{\sqrt{10}}$

But, $\sin\theta = \frac{d}{50} \Rightarrow \frac{d}{50} = \frac{3}{\sqrt{10}} \Rightarrow d = \frac{150}{\sqrt{10}} = 15\sqrt{10}$ km

2006 – Ordinary Level – Question 2

Ship A is travelling east $\alpha°$ north with a constant speed of $39\,\text{km/h}$, where $\tan\alpha = \frac{5}{12}$.
Ship B is travelling due east with a constant speed of $16\,\text{km/h}$.

At 2pm ship B is positioned 90 km due north of ship A.



(i) Express the velocity of ship A and the velocity of ship B in terms of $\vec{i}$ and $\vec{j}$.

(ii) Find the velocity of ship A relative to ship B in terms of $\vec{i}$ and $\vec{j}$.

(iii) Find the shortest distance between the ships.

Solution

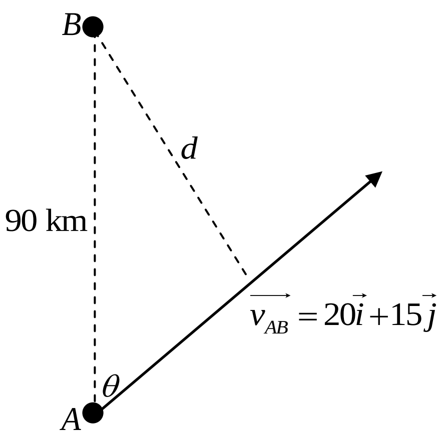
(i) $\overrightarrow{v_A} = 39\cos\alpha\vec{i} + 39\sin\alpha\vec{j}$ $\quad$ $\tan\alpha = \frac{5}{12} \Rightarrow \cos\alpha = \frac{12}{13}$ and $\sin\alpha = \frac{5}{13}$

$\Rightarrow$ $\overrightarrow{v_A} = 39\left(\frac{12}{13}\right)\vec{i} + 39\left(\frac{5}{13}\right)\vec{j} = 36\vec{i} + 15\vec{j}$

$\overrightarrow{v_B} = 16\vec{i}$

(ii) $\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B} = 36\vec{i} + 15\vec{j} - 16\vec{i} = 20\vec{i} + 15\vec{j}$

(iii)



$\tan\theta = \frac{20}{15} = \frac{4}{3} \Rightarrow \cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$

But, $\sin\theta = \frac{d}{90}$ $\quad\Rightarrow\quad \frac{d}{90} = \frac{4}{5}$

$\Rightarrow\quad d = 72$ km

2004 – Higher Level – Question 2(b)

At time $t=0$, two particles P and Q are set in motion.

At time $t=0$, Q has position vector $20\vec{i}+40\vec{j}$ metres relative to P.

P has a constant velocity of $3\vec{i}+5\vec{j}\,\text{m/s}$ and Q has a constant velocity of $4\vec{i}-3\vec{j}\,\text{m/s}$.

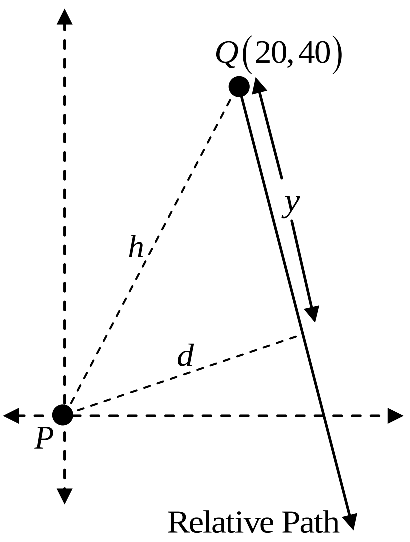
Find

(i) the velocity of Q relative to P

(ii) the shortest distance between P and Q, to the nearest metre

(iii) the time when P and Q are closest together, correct to one decimal place.

Solution



(i) $\overrightarrow{v_P}=3\vec{i}+5\vec{j}$ $\qquad$ $\overrightarrow{v_Q}=4\vec{i}-3\vec{j}$

$\overrightarrow{v_{QP}}=\overrightarrow{v_Q}-\overrightarrow{v_P}=4\vec{i}-3\vec{j}-\left(3\vec{i}+5\vec{j}\right)=\vec{i}-8\vec{j}$

(ii) Equation of Relative Path:

$y-y_1=m(x-x_1)$

$\Rightarrow \quad y-40=-8(x-20)$

$\Rightarrow \quad 8x+y-200=0$

$d=$ perpendicular distance from $(0,0)$ to relative path

$$\Rightarrow \quad d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}=\frac{|-200|}{\sqrt{65}}=\frac{200\sqrt{65}}{65}=\frac{40\sqrt{65}}{13}\simeq 25\text{ m}$$

(iii) $$\text{Time}=\frac{\text{Relative Distance}}{\text{Relative Speed}}=\frac{y}{\sqrt{65}}$$

$h=\sqrt{20^2+40^2}=\sqrt{2000}$

$y^2=h^2-d^2$ ...Pythagoras' Theorem

$$\Rightarrow \quad y=\sqrt{2000-\left(\frac{40\sqrt{65}}{13}\right)^2}=\frac{60\sqrt{65}}{13}$$

$$\Rightarrow \quad \text{Time}=\frac{\frac{60\sqrt{65}}{13}}{\sqrt{65}}=\frac{60}{13}\simeq 4\cdot 6\text{ seconds}$$

1996 – Higher Level – Question 2

A ship B, is travelling due West at $25\cdot6\,\text{km/h}$. A second ship, C, travelling at $32\,\text{km/h}$ is first sighted 17 km due north of B. From B the ship C appears to be moving South-east.

Find

(i) the direction in which C is actually moving

(ii) the velocity of C relative to B

(iii) the shortest distance between the ships in the subsequent motion

(iv) the time that elapses, after first sighting, before the ships are again 17 km apart.

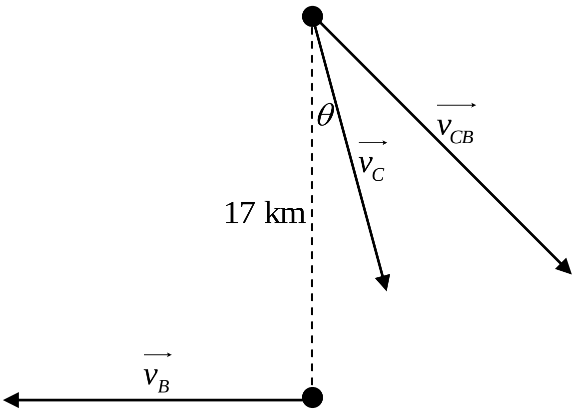
Solution

(i) $\overrightarrow{v_B} = -25\cdot6\vec{i}$

$\overrightarrow{v_{CB}} = x\cos 45°\vec{i} - x\sin 45°\vec{j} = \frac{x}{\sqrt{2}}\vec{i} - \frac{x}{\sqrt{2}}\vec{j}$

$\overrightarrow{v_C} = 32\sin\theta\vec{i} - 32\cos\theta\vec{j}$

$\overrightarrow{v_{CB}} = \overrightarrow{v_C} - \overrightarrow{v_B}$



$\Rightarrow \quad \frac{x}{\sqrt{2}}\vec{i} - \frac{x}{\sqrt{2}}\vec{j} = (32\sin\theta + 25\cdot6)\vec{i} - 32\cos\theta\vec{j}$

$\Rightarrow \quad 32\sin\theta + 25\cdot6 = 32\cos\theta$

$\Rightarrow \quad 32(\cos\theta - \sin\theta) = 25\cdot6$

$\Rightarrow \quad \cos\theta - \sin\theta = \frac{4}{5}$ ...divide by $\cos\theta$

$\Rightarrow \quad 1 - \tan\theta = \frac{4}{5}\sec\theta$

$\Rightarrow \quad 1 - \tan\theta = \frac{4}{5}\sqrt{1+\tan^2\theta}$

$\Rightarrow \quad 5 - 5\tan\theta = 4\sqrt{1+\tan^2\theta}$ ...square both sides

$\Rightarrow \quad 25 - 50\tan\theta + 25\tan^2\theta = 16 + 16\tan^2\theta$

$\Rightarrow \quad 9\tan^2\theta - 50\tan\theta + 9 = 0$

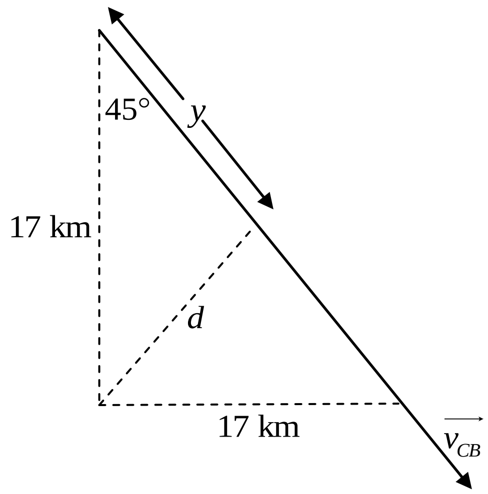
$\Rightarrow \quad \tan\theta = \frac{50 \pm \sqrt{2500-324}}{18} = \frac{50 \pm 8\sqrt{34}}{18} = \frac{25 \pm 4\sqrt{34}}{9}$

$\Rightarrow \quad$ ~~$\theta = 79\cdot4°$~~ $\quad \boxed{\theta = 10\cdot6°}$

(ii) $\frac{x}{\sqrt{2}} = 32\cos\theta = 31\cdot459 \quad \Rightarrow \quad \overrightarrow{v_{CB}} = 31\cdot459\vec{i} - 31\cdot459\vec{j}$

(iii) $\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{d}{17}$

$\Rightarrow \quad d = \frac{17}{\sqrt{2}} \simeq 12 \text{ km}$



(iv) $\text{Time} = \frac{\text{Relative Distance}}{\text{Relative Speed}} = \frac{2y}{44 \cdot 49}$ ... $44 \cdot 49$ is the magnitude of $\overrightarrow{v_{CB}}$

$$y^2 = 17^2 - \left(\frac{17}{\sqrt{2}}\right)^2$$

$\Rightarrow \quad y = \frac{17\sqrt{2}}{2} \quad \Rightarrow \quad 2y = 17\sqrt{2}$

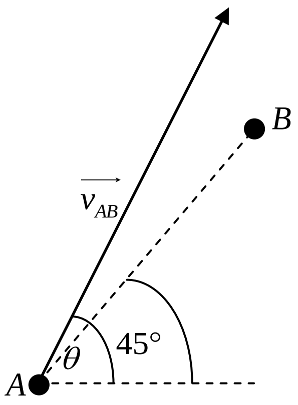
$\Rightarrow \quad \text{Time} = \frac{17\sqrt{2}}{44 \cdot 49} = 0 \cdot 54 \text{ hours} = 32 \text{ minutes } 24 \text{ seconds}$

2007 – Higher Level – Question 2(a)

Ship B is travelling west at $24\,\text{km/h}$. Ship A is travelling north at $32\,\text{km/h}$.
At a certain instant ship B is 8 km north-east of ship A.

(i) Find the velocity of ship A relative to ship B.

(ii) Calculate the length of time, to the nearest minute, for which the ships are less than or equal to 8 km apart.



Solution

(i) $\overrightarrow{v_A} = 32\vec{j}$ $\qquad \overrightarrow{v_B} = -24\vec{i}$

$\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B}$

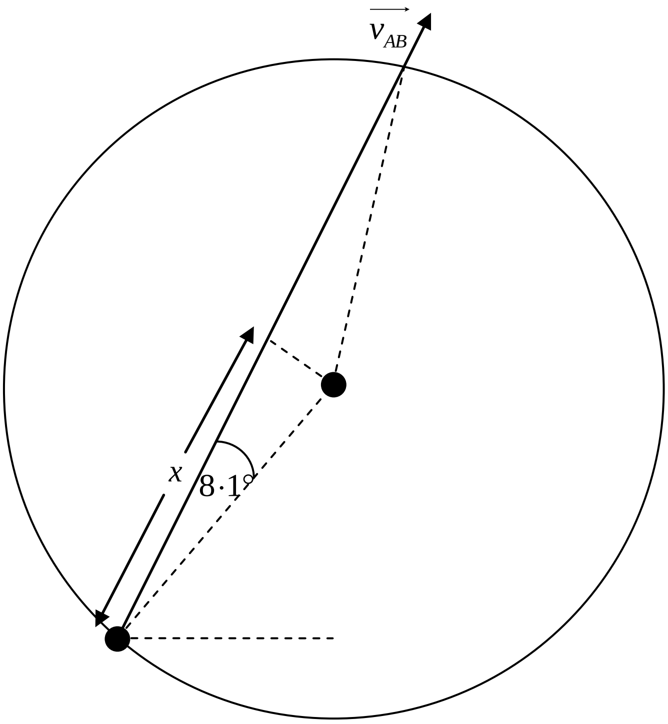
$\Rightarrow \quad \overrightarrow{v_{AB}} = 24\vec{i} + 32\vec{j} \qquad \left|\overrightarrow{v_{AB}}\right| = \sqrt{24^2 + 32^2} = 40\,\text{km/h}$

$\tan\theta = \dfrac{32}{24} = \dfrac{4}{3} \qquad \Rightarrow \qquad \theta = 53{\cdot}1^\circ$

(ii) $\text{Time} = \dfrac{\text{Relative Distance}}{\text{Relative Speed}} = \dfrac{2x}{40} = \dfrac{x}{20}$

$\cos 8{\cdot}1^\circ = \dfrac{x}{8} \qquad \Rightarrow \qquad x = 7{\cdot}92\text{ km}$

$\Rightarrow \quad \text{Time} = \dfrac{7{\cdot}92}{20} = 0{\cdot}396\text{ hours} = 23\text{ minutes }46\text{ seconds}$



2002 – Higher Level – Question 2(a)

Two boats, B and C, are each moving with constant velocity.
At a certain instant, boat B is 10 km due west of boat C.
The speed and direction of boat B relative to boat C is $2\cdot 5\,\text{m/s}$ in the direction $60°$ south of east.

(i) Calculate the shortest distance between the two boats, to the nearest metre.

(ii) Calculate the length of time, to the nearest second, for which the boats are less than or equal to 9 km apart.

Solution

(i) $\sin 60° = \frac{\sqrt{3}}{2} = \frac{d}{10}$

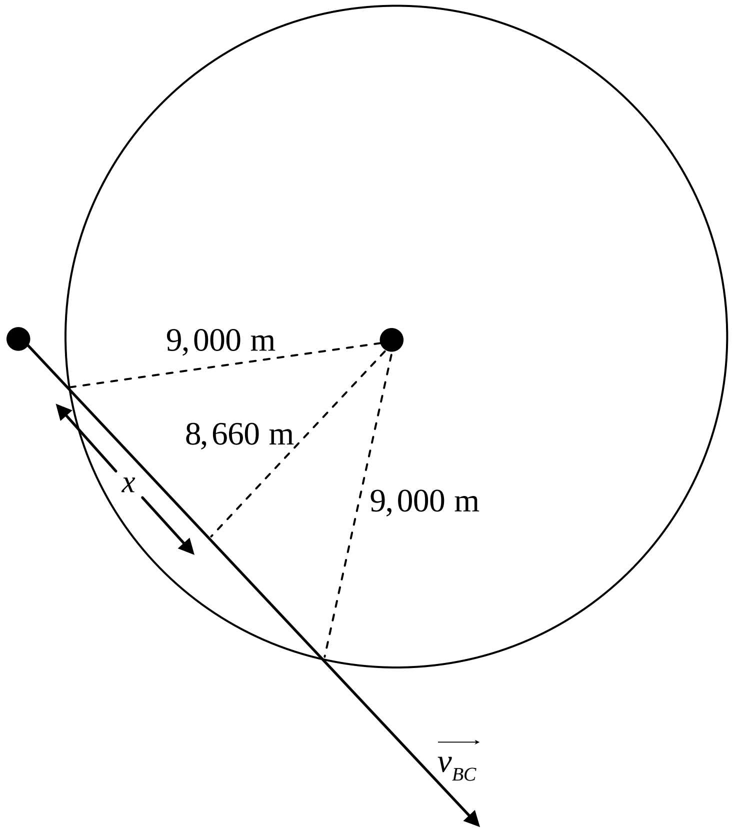
$\Rightarrow \quad d = 5\sqrt{3}\text{ km} \simeq 8,660\text{ m}$

(ii) $\text{Time} = \frac{\text{Relative Distance}}{\text{Relative Speed}} = \frac{2x}{2\cdot 5}$

$x^2 = 9,000^2 - 8,660^2$ ...Pythagoras' Theorem

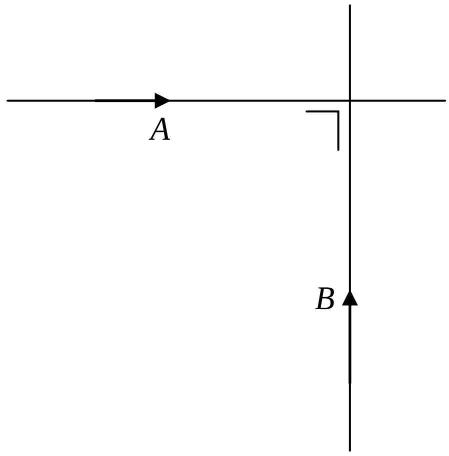
$\Rightarrow \quad x = 2,450\text{ m} \quad \Rightarrow \quad 2x = 4,900\text{ m}$

$\Rightarrow \quad \text{Time} = \frac{4,900}{2\cdot 5} = 1,960\text{ seconds}$



2009 – Higher Level – Question 2(a)

Two cars, A and B, travel along two straight roads which intersect at right angles.
A is travelling east at $15\,\text{m/s}$.
B is travelling north at $20\,\text{m/s}$.



At a certain instant both cars are 800 m from the intersection and approaching the intersection.

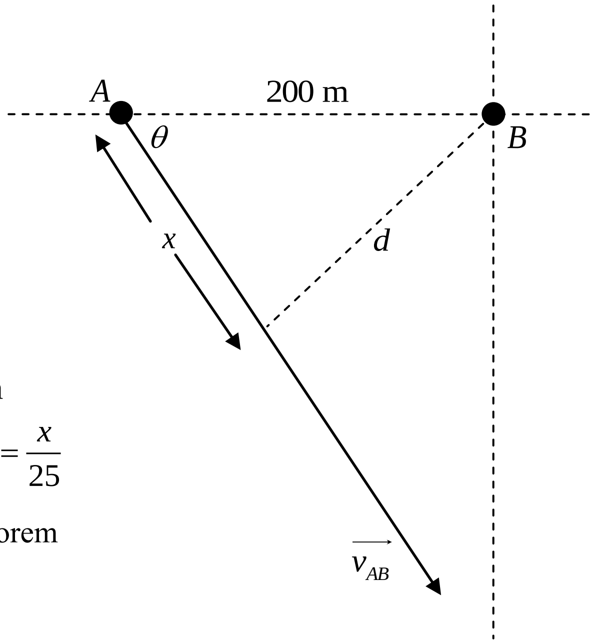
Find (i) the shortest distance between the cars
(ii) the distance each car is from the intersection when they are nearest to each other.

Solution

(i) Allow B to go to intersection.

$$\text{Time to intersection} = \frac{\text{Distance}}{\text{Speed}} = \frac{800}{20} = 40 \text{ seconds}$$

In the meantime, A has travelled $15 \times 40 = 600$ m and is now 200 m from intersection.



$\vec{v}_A = 15\vec{i}$

$\vec{v}_B = 20\vec{j}$

$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B = 15\vec{i} - 20\vec{j}$

$\tan\theta = \frac{20}{15} = \frac{4}{3} \quad \Rightarrow \quad \sin\theta = \frac{4}{5}$

But, $\sin\theta = \frac{d}{200} \quad \Rightarrow \quad \frac{d}{200} = \frac{4}{5}$

$\Rightarrow \quad d = 160$ m

(ii) $\text{Time} = \frac{\text{Relative Distance}}{\text{Relative Speed}} = \frac{x}{\sqrt{15^2 + 20^2}} = \frac{x}{25}$

$x^2 = 200^2 - 160^2$ ...Pythagoras' Theorem

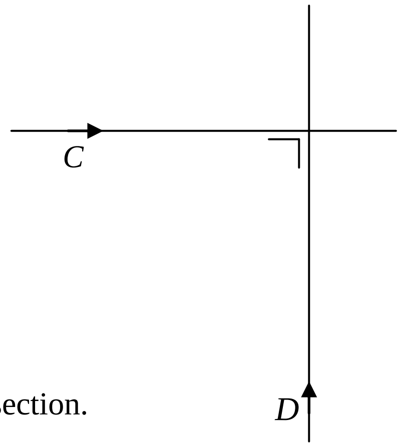
$\Rightarrow \quad x = 120$ m

$\Rightarrow \quad \text{Time} = \frac{120}{25} = 4 \cdot 8$ seconds

A has travelled $15 \times 4 \cdot 8 = 72$ m $\Rightarrow$ 128 m from intersection

B has travelled $20 \times 4 \cdot 8 = 96$ m $\Rightarrow$ 96 m from intersection.

2008 – Higher Level – Question 2(a)

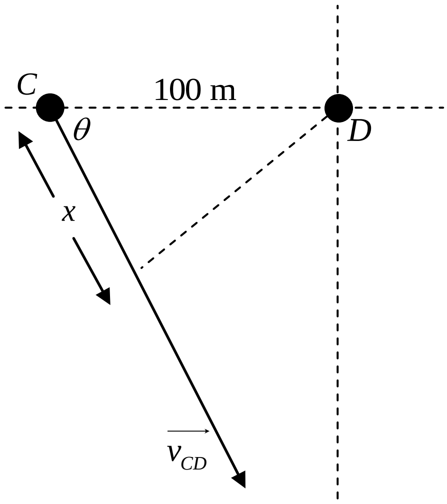


Two straight roads cross at right angles.
A woman C, is walking towards the intersection with a uniform speed of $1 \cdot 5\,\text{m/s}$.
Another woman D is moving towards the intersection with a uniform speed of $2\,\text{m/s}$.

C is 100 m away from the intersection as D passes the intersection.

Find (i) the velocity of C relative to D
(ii) the distance of C from the intersection when they are nearest together.

Solution



(i) $\overrightarrow{v_C} = 1 \cdot 5\vec{i}$ $\qquad$ $\overrightarrow{v_D} = 2\vec{j}$

$\overrightarrow{v_{CD}} = \overrightarrow{v_C} - \overrightarrow{v_D} = 1 \cdot 5\vec{i} - 2\vec{j}$

(ii) $\tan\theta = \dfrac{2}{1 \cdot 5} = \dfrac{4}{3} \quad \Rightarrow \quad \cos\theta = \dfrac{3}{5}$

But, $\cos\theta = \dfrac{x}{100} \quad \Rightarrow \quad \dfrac{x}{100} = \dfrac{3}{5}$

$\Rightarrow \quad x = 60\text{ m}$

$\text{Time} = \dfrac{\text{Relative Distance}}{\text{Relative Speed}} = \dfrac{60}{\sqrt{1 \cdot 5^2 + 2^2}}$

$\Rightarrow$ $\text{Time} = 24\text{ seconds}$

C has travelled $1 \cdot 5 \times 24 = 36\text{ m}$ $\quad \Rightarrow \quad$ C is now 64 m from intersection.

2009 – Higher Level – Question 2(b)

The speed of an aeroplane in still air is u km/h.
The aeroplane flies a straight line course from P to Q, where Q is north of P.

If there is no wind blowing the time for the journey from P to Q is T hours.

Find, in terms of u and T, the time to fly from P to Q if there is a wind blowing from the south-east with a speed of $4\sqrt{2}$ km/h.

Solution

No Wind

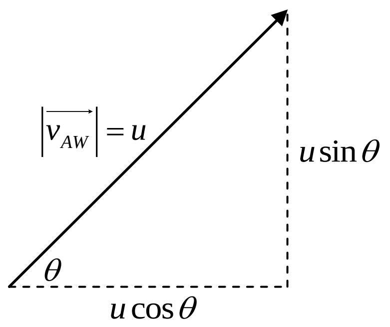
$\overrightarrow{v_A} = u$

$T = \dfrac{\text{Distance}}{\text{Speed}} = \dfrac{|PQ|}{u}$

$\Rightarrow \quad \boxed{|PQ| = uT}$ ...actual distance from P to Q.

$\overrightarrow{v_W} = -4\sqrt{2}\cos 45°\vec{i} + 4\sqrt{2}\sin 45°\vec{j}$

$\boxed{\overrightarrow{v_W} = -4\vec{i} + 4\vec{j}}$



$\overrightarrow{v_{AW}} = u\cos\theta\vec{i} + u\sin\theta\vec{j}$

$\overrightarrow{v_{AW}} = \overrightarrow{v_A} - \overrightarrow{v_W}$

$\Rightarrow \quad \overrightarrow{v_A} = \overrightarrow{v_{AW}} + \overrightarrow{v_W}$

$\Rightarrow \quad \overrightarrow{v_A} = (u\cos\theta - 4)\vec{i} + (u\sin\theta + 4)\vec{j}$

$\overrightarrow{v_A}$ needs to be due north $\quad \Rightarrow \quad \vec{i} - \text{component} = 0$

$\Rightarrow \quad u\cos\theta - 4 = 0 \Rightarrow \quad \cos\theta = \dfrac{4}{u} \quad \Rightarrow \quad \sin\theta = \dfrac{\sqrt{u^2 - 16}}{u}$

$\Rightarrow \quad \overrightarrow{v_A} = \left(\cancel{u}\left[\dfrac{\sqrt{u^2-16}}{\cancel{u}}\right] + 4\right)\vec{j} = \left(4 + \sqrt{u^2 - 16}\right)\vec{j}$

$\Rightarrow \quad |\overrightarrow{v_A}| = 4 + \sqrt{u^2 - 16}$

$\text{Time} = \dfrac{\text{Distance}}{\text{Speed}}$

$\Rightarrow \quad \boxed{\text{Time} = \dfrac{uT}{4 + \sqrt{u^2 - 16}}}$

2006 – Higher Level – Question 2(a)

Two aeroplanes A and B, moving horizontally, are travelling at $200\,\text{km/h}$ relative to the ground. There is a wind blowing from the east at $60\,\text{km/h}$. The actual directions of flight A and B are north-west and north-east respectively.

Find (i) the speed of aeroplane A in still air
(ii) the magnitude and direction of the velocity of A relative to B.

Solution

(i) $\overrightarrow{v_W} = -60\vec{i}$

$\overrightarrow{v_A} = -200\cos 45°\vec{i} + 200\sin 45°\vec{i} = -100\sqrt{2}\,\vec{i} + 100\sqrt{2}\,\vec{j}$

$\overrightarrow{v_B} = 100\sqrt{2}\,\vec{i} + 100\sqrt{2}\,\vec{j}$

Speed of Aeroplane A in still air $= \overrightarrow{v_{AW}}$

$\overrightarrow{v_{AW}} = \overrightarrow{v_A} - \overrightarrow{v_W}$

$\Rightarrow \quad \overrightarrow{v_{AW}} = -100\sqrt{2}\,\vec{i} + 100\sqrt{2}\,\vec{j} + 60\vec{i}$

$\Rightarrow \quad \overrightarrow{v_{AW}} = \left(60 - 100\sqrt{2}\right)\vec{i} + 100\sqrt{2}\,\vec{j}$

$\left|\overrightarrow{v_{AW}}\right| = \sqrt{\left(60 - 100\sqrt{2}\right)^2 + \left(100\sqrt{2}\right)^2} = 163{\cdot}19\,\text{km/h}$

(ii) $\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B}$

$\overrightarrow{v_{AB}} = -100\sqrt{2}\,\vec{i} + \cancel{100\sqrt{2}\,\vec{j}} - \left(100\sqrt{2}\,\vec{i} + \cancel{100\sqrt{2}\,\vec{j}}\right)$

$\overrightarrow{v_{AB}} = -200\sqrt{2}\,\vec{i}$

$\left|\overrightarrow{v_{AB}}\right| = 200\sqrt{2}$

Direction: Due West.

2008 – Higher Level – Question 2(b)

On a particular day the velocity of the wind, in terms of $\vec{i}$ and $\vec{j}$, is $x\vec{i}-3\vec{j}$, where $x\in\mathbb{N}$.

$\vec{i}$ and $\vec{j}$ are unit vectors in the directions East and North respectively.

To a man travelling due East the wind appears to come from a direction North $\alpha°$ West where $\tan\alpha=2$.

When he travels due North at the same speed as before, the wind appears to come from a direction North $\beta°$ West where $\tan\beta=\frac{3}{2}$.

Find the actual direction of the wind.

Solution

Travelling East

$\overrightarrow{v_M}=p\vec{i}$

$\overrightarrow{v_W}=x\vec{i}-3\vec{j}$

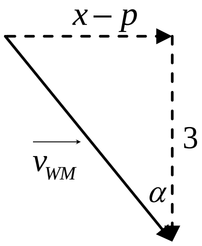
$\overrightarrow{v_{WM}}=\overrightarrow{v_W}-\overrightarrow{v_M}$

$\overrightarrow{v_{WM}}=(x-p)\vec{i}-3\vec{j}$

$\tan\alpha=\frac{x-p}{3}$

$2=\frac{x-p}{3}$

$p=x-6$



Travelling North

$\overrightarrow{v_M}=p\vec{j}$

$\overrightarrow{v_W}=x\vec{i}-3\vec{j}$

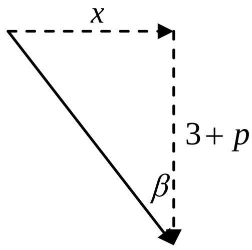
$\overrightarrow{v_{WM}}=\overrightarrow{v_W}-\overrightarrow{v_M}$

$\overrightarrow{v_{WM}}=x\vec{i}-(3+p)\vec{j}$

$\tan\beta=\frac{x}{3+p}$

$\frac{3}{2}=\frac{x}{3+p}$

$p=\frac{2x-9}{3}$

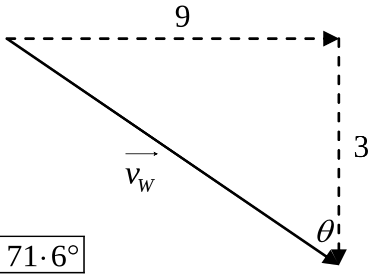


$\Rightarrow \quad x-6=\frac{2x-9}{3}$

$\Rightarrow \quad 3x-18=2x-9$

$\Rightarrow \quad x=9$

$\Rightarrow \quad \overrightarrow{v_W}=9\vec{i}-3\vec{j}$



$\tan\theta=3 \quad \Rightarrow \quad \boxed{\theta=71\cdot6°}$

2010 – Higher Level – Question 2(b)

When a motor-cyclist travels along a straight road from South to North at a constant speed of $12\cdot5$ ms^{-1} the wind appears to come from a direction North $45°$ East.

When she returns along the same road at the same constant speed, the wind appears to come from a direction South $45°$ East.

Find the magnitude and direction of the velocity of the wind.

Solution

South to North

$\overrightarrow{v_M} = 12\cdot5\vec{j}$

$\overrightarrow{v_{WM}} = -p\vec{i} - p\vec{j}$

$\overrightarrow{v_W} = \overrightarrow{v_{WM}} + \overrightarrow{v_M}$

$\overrightarrow{v_W} = -p\vec{i} + (12\cdot5 - p)\vec{j}$

North to South

$\overrightarrow{v_M} = -12\cdot5\vec{j}$

$\overrightarrow{v_{WM}} = -q\vec{i} + q\vec{j}$

$\overrightarrow{v_W} = \overrightarrow{v_{WM}} + \overrightarrow{v_M}$

$\overrightarrow{v_W} = -q\vec{i} + (q - 12\cdot5)\vec{j}$

$$\overrightarrow{v_W} = \overrightarrow{v_W}$$

$$\Rightarrow \quad -p\vec{i} + (12\cdot5 - p)\vec{j} = -q\vec{i} + (q - 12\cdot5)\vec{j}$$

$$\Rightarrow \quad p = q \quad \text{and} \quad 12\cdot5 - p = q - 12\cdot5$$

$$\Rightarrow \quad p = q = 12\cdot5$$

$$\Rightarrow \quad \overrightarrow{v_W} = -12\cdot5\vec{i} + 0\vec{j}$$

$$\left|\overrightarrow{v_W}\right| = 12\cdot5\,\text{m}/s$$

Direction: Due West